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THE MOTION OF MAGNETIZABLE FLUID IN A ROTATING HOMOGENEOUS MAGNETIC FIELD

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A general steady solution of cylindrically symmetric form is obtained for equations that define the motion of a magnetizable incompressible nonconducting fluid in a homogeneous magnetic field rotating at constant angular velocity. The obtained solution is used for determining the velocity and internal moment of momentum of a magnetizable fluid between two coaxial cylinders rotating at different angular velocities. It is shown that in the case of fixed cylinders a fluid posessing an internal moment of momentum is in motion, unlike the conventional viscous fluid. Various limit cases are considered when either the inner or outer cylinder is absent, when one of the cylinder is fixed and the other freely rotates, and when the two cylinders freely rotate.

Finely dispersed suspensions of ferromagnetic particles represent an example of magnetizable fluids. Orderly rotation of such particles (induced, for instance, by a rotating magnetic field) creates in the fluid an internal moment of momentum. The equation of variation of the internal moment of momentum was given in a general form in the monograph [1].

Equations which define ferromagnetic suspensions when the internal moment of momentum can be neglected were derived in [2, 3]. Here ferromagnetic suspensions are considered to be monophase media, and the presence of ferromagnetic particles is taken into account by the introduction of the internal moment of momentum of a unit volume and the magnetization intensity of suspension which is related to the number of particles, their relative disposition, and individual particle magnetization intensity. The equation defining the motion of such medium without allowance for the hydro magnetic effect was first derived in [4].

1. General solution of equations defining magnetizable fluid with internal rotation in the case of cylindrical symmetry. A closed system of equations defining the motion of a magnetizable fluid with allowance for the internal moment of momentum was derived in [4]. When the medium is nonconducting the incompressible the phenomenological coefficients are independent of the magnetic field. These equations without allowance for cross effects ferrohydrodynamics approximation [2] are of the form

$$\operatorname{div}\mathbf{u} = 0, \quad \rho\frac{d\mathbf{u}}{dt} = -\nabla p + \mu\Delta\mathbf{u} + \frac{1}{2\tau_s}\operatorname{rot}(\mathbf{K} - I\boldsymbol{\Omega}) + (\mathbf{M}\nabla)\mathbf{H} \tag{1.1}$$

$$\frac{d\mathbf{K}}{dt} = \delta\Delta\mathbf{K} + \delta_1\nabla\operatorname{div}\mathbf{K} - \frac{1}{\tau_s}(\mathbf{K} - I\boldsymbol{\Omega}) + [\mathbf{M}\times\mathbf{H}]$$

$$\frac{d\mathbf{M}}{dt} = I^{-1}[\mathbf{K}\times\mathbf{M}] - \frac{\mathbf{M}-\chi\mathbf{H}}{\tau}, \quad \operatorname{div}(\mathbf{H}+4\pi\mathbf{M}) = 0, \quad \operatorname{rot}\mathbf{H} = 0$$

where $\mathbf{u}$, ρ, p, $\mathbf{K}$ and $\mathbf{M}$ are, respectively, the velocity, density, pressure, internal moment of momentum of a unit volume, and the medium magnetization intensity; $\mathbf{H}$ is the magnetic field; $\mathbf{\Omega} = \operatorname{rot}\mathbf{u}/2$ is the vortex vector, and μ, τ_s, δ, δ_1, I and τ are phenomenological coefficients which here are assumed constant. We assume that the magnetic permeability coefficient χ is independent of temperature. In this case the energy equation is separated from the system of Eqs. (1.1) that defines the motion of incompressible fluid and is not considered here.

We assume for simplicity that $4\pi M \ll H$. This inequality is confirmed by experiments. It can be assumed that throughout the fluid the magnetic field is equal to the external homogeneous field. The last two of Eqs.(1.1) are identically satisfied.

We shall consider the steady motion of fluid in a homogeneous magnetic field rotating in some plane at constant angular velicity Ω_f. The motion is assumed to be cylindrically symmetric about some axis perpendicular to the plane of field rotation. We select the system of cylindrical coordinates r, φ, z so that the z-axis is the axis of symmetry. We assume that $K_\varphi = K_r = 0$, $K_z = K$, $u_r = u_z = 0$, $u_\varphi = u$, $\Omega_{f\varphi} = \Omega_{fr} = 0$, and $\Omega_{fz} = \Omega_f$ and that all functions depend only on r, i.e. that $\partial/\partial t = \partial/\partial\varphi = \partial/\partial z = 0$. The stable solution of the fourth equation of system (1.1) is of the form

$$M_\parallel = \frac{I^2\chi H}{I^2+\tau^2(K-I\Omega_f)^2}, \quad M_\perp = \frac{\chi\tau I H(I\Omega_f - K)}{I^2+\tau^2(K-I\Omega_f)^2} \qquad (1.2)$$

where $M_\parallel$ and $M_\perp$ are projections of the magnetization intensity vector $\mathbf{M}$ on the direction of the magnetic field $\mathbf{H}$ and on the direction normal to the magnetic field vector, respectively, and H is the absolute intensity of the magnetic field.

Using solution (1.2) of the equation of magnetization intensity we obtain

$$[\mathbf{M}\times\mathbf{H}]_z = M_\perp H = \frac{\chi\tau I H^2(I\Omega_f - K)}{I^2+\tau^2(K-I\Omega_f)^2} \qquad (1.3)$$

The equations of motion and moment of momentum (the second and third equations of system (1.1) projected on the coordinate axes φ and z, respectively), with allowance for equality (1.3) and assumptions made above, are of the form

$$\mu\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}\right)-\frac{1}{2\tau_s}\frac{d}{dr}(K-I\Omega) = 0, \qquad (1.4)$$

$$\Omega = \frac{1}{2}\left(\frac{du}{dr}+\frac{u}{r}\right)$$

$$\delta\left(\frac{d^2K}{dr^2}+\frac{1}{r}\frac{dK}{dr}\right)-\frac{1}{\tau_s}(K-I\Omega)+M_\perp H = 0$$

Integration of the first equation of system (1.4) yields

$$\Omega = \frac{\gamma(K+C_1')}{I}, \quad \gamma = \frac{1}{1+2R}, \quad R = \frac{2\mu\tau_s}{I}, \quad C_1' = \text{const} \qquad (1.5)$$

We seek the general solution of Eqs. (1.1) on the assumption that $\tau^2\Omega_f^2 \ll 1$. The system (1.4) with allowance for formulas (1.5) and (1.2) has a general solution of the form

$$\frac{K}{I\Omega_f} = \frac{Q+C_1}{Q+1} + C_2 I_0\left(\frac{r}{l}\right) + C_3 K_0\left(\frac{r}{l}\right) \tag{1.6}$$

$$C_1 = \frac{\gamma C_1'}{I\Omega_f} + \gamma \frac{I\Omega_f Q + \gamma C_1'}{I\Omega_f (Q+1-\gamma)}$$

$$\frac{u}{\Omega_f} = C_1 r + \frac{C_4}{r} + 2\gamma l C_2 I_1\left(\frac{r}{l}\right) - 2\gamma l C_3 K_1\left(\frac{r}{l}\right), \quad l^2 = \frac{\delta\tau_s}{Q+1-\gamma}$$

$$Q = \chi\tau\tau_s I^{-1} H^2$$

where $I_n(r/l)$ and $K_n(r/l)$ are modified Bessel and Hankel functions of order n, respectively. Constants of integration C_1, C_2, C_3, and C_4 are determined by boundary conditions.

2. Motion of a magnetizable fluid between two cylinders in rotating magnetic field. Let us consider the flow of fluid between two coaxial cylinders of radius R_1 and R_2, $R_1 < R_2$, rotating around their axis at angular velocities Ω_1 and Ω_2 (here and in what follows subscripts 1 and 2 relate to the inner and outer cylinder, respectively). The cylinder axis is normal to the plane of the magnetic field rotation. As the boundary conditions we take the conditions of sticking at the solid boundary for the velocity and moment of momentum of motion of a viscous ferro fluid. These conditions are of the form

$$\mathbf{u}(R_i) = \Omega_i R_i, \quad \mathbf{K}(R_i) = I\Omega_i, \quad i = 1, 2 \tag{2.1}$$

Conditions (2.1) yield a system of linear equations for the determination of constants C_1, C_2, C_3, and C_4 that appear in formulas (1.6). Solving that system we obtain

$$C_1 = c(Q+1) - Q - a(Q+1)C_2 + b(Q+1)C_3 \tag{2.2}$$

$$C_2 = \Delta^{-1}\{[K_0(R_2/l) + b](\omega_1 - c) - [K_0(R_1/l) + b] \times (\omega_2 - c)\}$$

$$C_3 = \Delta^{-1}\{[I_0(R_1/l) - a](\omega_2 - c) - [I_0(R_2/l) - a](\omega_1 - c)\}$$

$$C_4 = R_1^2(\omega_1 - C_1) - 2\gamma l R_1 I_1(R_1/l) C_2 + 2\gamma l R_1 K_1(R_1/l) C_3$$

$$a = 2\gamma l \frac{R_1 I_1(R_1/l) - R_2 I_1(R_2/l)}{(R_1^2 - R_2^2)(Q+1)}$$

$$b = 2\gamma l \frac{R_1 K_1(R_1/l) - R_2 K_1(R_2/l)}{(R_1^2 - R_2^2)(Q+1)}$$

$$c = \frac{Q}{Q+1} + \frac{\omega_1 R_1^2 - \omega_2 R_2^2}{(R_1^2 - R_2^2)(Q+1)}, \quad \omega_i = \frac{\Omega_i}{\Omega_f}, \quad i = 1, 2$$

$$\Delta = [I_0(R_1/l) - a][K_0(R_2/l) + b] - [I_0(R_2/l) - a][K_0(R_1/l) + b]$$

Passing in fomulas (1.6) and (2.2) to the limit $R_2 \to \infty$, in the case when the two cylinders are stationary ($\omega_1 = \omega_2 = 0$) we obtain

$$\frac{K}{I\Omega_f} = c\left[1 - \frac{K_0(r/l)}{K_0(R_1/l)}\right], \quad c = \frac{Q}{Q+1} \tag{2.3}$$

$$\frac{u}{\Omega_f} = 2\gamma l \frac{c}{K_0(R_1/l)}\left[K_1\left(\frac{r}{l}\right) - \frac{R_1 K_1(R_1/l)}{r}\right]$$

Solution (2.3) defines the flow of a magnetizable fluid outside a stationary cylinder. Analysis of this solution shows that the flow outside a stationary cylinder is of opposite direction to that of field rotation.

Passing in formulas (1.6) and (2.2) to the limit $R_1 \to 0$, for the flow inside a stationary cylinder we obtain

$$\frac{K}{I\Omega_f} = c - \frac{c}{I_0(R_2/l) - a}\left[I_0\left(\frac{r}{l}\right) - a\right], \quad a = \frac{2\gamma l I_1(R_2/l)}{R_2} \tag{2.4}$$

$$\frac{u}{\Omega_f} = -\frac{2\gamma l c}{I_0(R_2/l) - a}\left[I_1\left(\frac{r}{l}\right) - \frac{r}{R_2} I_1\left(\frac{R_2}{l}\right)\right]$$

Formulas (2.4) were first obtained in [5, 6]. Their analysis shows that inside a stationary cylinder the flow rotates in the same direction as the field. This result is in qualitative agreement with experimental data [7, 8].

The stationary cylinders of radii R_1 and R_2 are subjected to moments $\mathbf{M}_1 = M_1\mathbf{e}$, and $\mathbf{M}_2 = M_2\mathbf{e}$ ($\mathbf{e}$ is the unit vector of the z-axis) induced by the magnetizable fluid forces calculated per unit height of cylinder. They are associated with the presence of surface friction and of the internal moment of momentum

$$M_i = (2\pi R_i^2 p_{r\varphi} + 2\pi R_i Q_{zr})|_{r=R_i} \cdot n_i, \quad n_1 = 1, \quad n_2 = -1 \tag{2.5}$$

$$Q_{zr} = \delta \frac{dK}{dr}, \quad p_{r\varphi} = \mu\left(\frac{du}{dr} - \frac{u}{r}\right) - \frac{1}{2\tau_s}(K - I\Omega)$$

where $p_{r\varphi}$ are stress tensor components acting in the magnetizable medium, and Q_{zr} is the component of the tensor of the internal moment of momentum flow of the medium. In the absence of an external cylinder moment M_1 of forces acting on the inner cylinder is

$$M_1 = -2\pi R_1 \frac{Q\Omega_f}{Q+1}\left[\frac{R_1 I}{2\tau_s} - \frac{\delta I K_1(R_1/l)}{l K_0(R_1/l)}\right] \tag{2.6}$$

and in the absence of an internal cylinder moment M_2 of forces acting on the internal cylinder is

$$M_2 = \pi R_2 Q I \Omega_f \frac{R_2 I_0(R_2/l) + 2(Q+\gamma) l I_1(R_2/l)}{\tau_s (Q+1)(I_0(R_2/l) - a)} \tag{2.7}$$

Let us consider the flow of magnetizable fluid between two coaxial cylinders one of which (for instance, the external) is fixed or rotates at specified angular velocity, while the other is free.

The flow of fluid is defined by formulas (1.6) with constants C_1, C_2, C_3, and C_4 determined by the following boundary conditions at the free cylinder and at the cylinder rotating at the specified angular velocity Ω_2:

$$\left\{\mu\left(\frac{du}{dr} - \frac{u}{r}\right) - \frac{1}{2\tau_s}(K - I\Omega)\right\}\Big|_{r=R_1} = 0, \quad \frac{dK}{dr}\Big|_{r=R_1} = 0 \tag{2.8}$$

$$u|_{r=R_2} = \Omega_2 R_2, \quad K|_{r=R_2} = I\Omega_2$$

The first two conditions of (2.8) are corollaries of the laws of conservation of the momentum and moment of momentum flows at the free surface, while the last two of these are conditions of sticking at the solid wall.

Substituting solutions (1.6) into boundary conditions (2.8) we obtain for constants C_1, C_2, C_3, and C_4 a system of linear equations.

In the absence of the external cylinder $(\omega_2 = 0,\ R_2 \to \infty)$ Eqs. (2.8) are simplified, and for the velocity of fluid outside the free cylinder and for the angular velocity Ω_c of the rotating cylinder we obtain

$$u = -\frac{Q\Omega_f R_1^2}{2(Q+1)Rr}, \quad \Omega_c = -\frac{Q\Omega_f}{2(Q+1)R}, \quad R = \frac{2\mu\tau_s}{I} \qquad (2.9)$$

The flow of magnetizable fluid between two coaxial cylinders of which the inner is free was earlier considered on the assumption that $\tau_s = 0$ and $\delta = 0$ [9] (*).

The solution of the free cylinder problem in infinite magnetizable fluid (formula (2.9)) implies that the cylinder must rotate in opposite direction to the rotation of the field.

Let us consider the case when both the inner and outer cylinders are free. The fluid flow is defined by formulas (1.6) with constant C_1, C_2, C_3, and C_4 that are determined by conditions at the free surfaces of cylinders. We have

$$u = \Omega_f r, \quad K = I\Omega_f \qquad (2.10)$$

Equalities (2.10) show that the complete system consisting of two free cylinders and fluid between them rotates as a solid body at angular velocity equal to that of the field rotation. This is in agreement with experimental data obtained in the Ferrohydrodynamics Laboratory of the Stavropol Teaching Institute.

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